MATH20132 Calculus of Several Variables. 2020-21

Problems 9 Lagrange's Method

1. For $\mathbf{x} \in \mathbb{R}^2$ let $f(\mathbf{x}) = x^2 - 3xy + y^2 - 5x + 5y$

i. Find the critical values of $f(\mathbf{x})$ in $\mathbb{R}^2$,

ii. Find the critical values of $f(\mathbf{x})$ restricted to the parametric curve $(t^2, t^3)^T$, $t \in \mathbb{R}$,

iii. Find the critical values of $f(\mathbf{x})$ restricted to the level set $x + 6y = 6$ (use Lagrange's method).

2. i. Find the minimum value of $3x^2 + 3y^2 + z^2$ subject to the condition $x + y + z = 1$.

ii. Find the maximum and minimum values of xy subject to the condition $x^2 + y^2 = 1$.

iii. Find the minimum and maximum values of xy^2 subject to the condition $x^2/a^2 + y^2/b^2 = 1$ (where a and b are positive constants).

3 Find points on the circle $(x-2)^2 + (y+1)^2 = 4$ which are a maximum and minimum distance from the origin.

Hint consider the **square** of the distance.

4. Find the minimum distance from the point on the x-axis $(a, 0)^T \in \mathbb{R}^2$ to the parabola $y^2 = x$.

5. Find the extremal values of $f(\mathbf{x}) = xy + yz$, $\mathbf{x} \in \mathbb{R}^3$ on the level set

$$\begin{aligned} x^2 + y^2 &= 1 \\ yz - x &= 0. \end{aligned}$$

6. Find the maximum and minimum values of $4y - 2z$ subject to the conditions $2x - y - z = 2$ and $x^2 + y^2 = 1$.

7 Find the minimum distance between a point on the circle in $\mathbb{R}^2$ with the equation $x^2 + y^2 = 1$ and a point on the parabola in $\mathbb{R}^2$ with the equation $y^2 = 2(4 - x)$.

8. An ellipse in $\mathbb{R}^3$ is given by the equations

$$\begin{cases} 2x^2 + y^2 = 4, \\ x + y + z = 0. \end{cases}$$

The intersection of a cylinder with a plane.

Use the method of Lagrange multipliers to find the points on the ellipse which are closest to the y-axis.

(This is a question from the June 2012 examination which turned out to be too difficult! It should be alright away from the pressure of the examination room. When you come to solving a system of equations remember to focus on finding x, y and z, i.e. remove the Lagrange parameters λ and μ as soon as possible.)

Additional Questions

Solutions have not been written up for all of the following.

9 Show that xy has a maximum on the ellipse $9x^2 + 4y^2 = 36$ and find it's value.

10 Find the maximum and minimum values of

$$x^2 + y^2 + z^2 - xy - xz - yz$$

subject to the condition

$$x^2 + y^2 + z^2 - 2x + 2y + 6z + 9 = 0.$$

11. Find the shortest distance from the origin to $x^2 + 3xy + y^2 = 4$.

12. Find the shortest distance from $(0, 0, 1)^T$ to $y^2 + x^2 + 4xy = 4$ in the x-y plane.

13. A cylindrical can (with top and bottom) has volume V. Subject to this constraint, what dimensions give it the least surface area?

Idea of solution If the cylinder of height h and radius r the area is $2\pi rh + 2\pi r^2$ and volume $\pi r^2 h$. So the essence of the question is to minimise $rh + r^2$ subject to $\pi r^2 h = V$.

14. Find the nearest point on the ellipse $x^2 + 2y^2 = 1$ to the line $x + y = 4$.

Idea of solution If $(x, y)^T$ is a point on the ellipse and $(u, v)^T$ a point on the line then $(x - u)^2 + (y - v)^2$ is the square of the distance between the two points. So need to minimise $(x - u)^2 + (y - v)^2$ subject to $x^2 + 2y^2 = 1$ and $u + v = 4$.

15. How close does the intersection of the planes $v + w + x + y + z = 1$ and $v - w + 2x - y + z = -1$ in $\mathbb{R}^5$ come to the origin?

Idea of solution To minimise $v^2 + w^2 + x^2 + y^2 + z^2$ (the square of the distance of $(v, w, x, y, z)^T$ from the origin) subject to $v + w + x + y + z = 1$ and $v - w + 2x - y + z = -1$. The answer is $\sqrt{612}/36$.

16. Let $x_1, ..., x_5$ be 5 positive numbers. Maximise their product subject to the constraint that $x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 300$.

17. Find the distance from the point $(10, 1, -6)$ to the intersection of the planes $x + y + 2z = 5$ and $2x - 3y + z = 12$.

18. If a and b are positive numbers find the maximum and minimum values of $(xv - yu)^2$ subject to the constraints $x^2 + y^2 = a^2$ and $u^2 + v^2 = b^2$.

19. Find the dimensions of the box parallel to the axes of maximum volume given that the surface area is $10m^2$.

Idea of solution If x, y and z are the lengths of the sides of the box then the volume is xyz and the surface area $2\,(xy + yz + xz)$. So maximise xyz subject to $xy + yz + xz = 5$.